APPLIED AND COMPUTATIONAL HARMONIC ANALYSIS **3,** 164–185 (1996)
ARTICLE NO. 0014

Implementation of Operators via Filter Banks

Hardy Wavelets and Autocorrelation Shell

G. Beylkin

Program in Applied Mathematics, University of Colorado, Boulder, Colorado 80304

and

B. Torrésani

CPT, CNRS-Luminy, Case 907, 13288 Marseille Cedex 09, France

Communicated by L. F. Greengard

written as

$$\hat{\psi}(\xi) = m_1(\xi/2)\,\hat{\phi}(\xi/2),$$

where $\hat{\psi}$ and $\hat{\phi}$ are the Fourier transform of the wavelet and of the scaling function. The 2π-periodic square-integrable function m_1 represents one of the QMFs. Our approach is based on the observation that if the wavelet $\psi(x)$ is sufficiently well localized in the Fourier domain, one may write

$$(\widehat{T\psi})(\xi) \approx m_T(\xi/4)\,\hat{\phi}(\xi/4), \qquad (1.1)$$

where m_T is a 2π-periodic function which is computed given the symbol of the operator T. The accuracy of the approximation in (1.1) is controlled by the number of vanishing moments of the wavelet ψ (it might be necessary to consider (1.1) on each scale separately if the symbol of T is not homogeneous). As a result, the operator T is implemented using filters m_T (may be different on different scales), where m_T plays a role similar to that of the filter m_1 of the QMF pair. The major difference, which is already visible in Eq. (1.1), is that the m_T filter performs a scaling by a factor 4 instead of 2. This has some practical implications, which are discussed throughout the paper. The factor 4 may be replaced by a factor $2^n, n \geq 2$, as a way to improve accuracy. In this way, the procedure for design of these subband filters allows us to attain any desired accuracy.

The approach of this paper (as that of [3]) may be traced back to the Calderón–Zygmund and Littlewood–Paley approaches to harmonic analysis of functions and operators which we apply here to design filters given the symbol of an operator. Our method may be used with any wavelets (associated with quadrature mirror filters) which possess a sufficient number of vanishing moments. In cases where the associated scaling function also has vanishing moments (which implies that the corresponding coefficients are well approximated by samples on fine scales), our algorithm leads to a fast method for computing the Hilbert transform (and, thus, modulus and phase) of signals. This is the case for the autocorrelation wavelets derived in [20] which we consider here in some detail. Although our approach is quite general, we concentrate on several specific examples, such as the Hilbert transform, operators of differentiation, and, more generally, convolution operators. In particular, we consider the Hilbert transform both as an example and as an important special case, because of its particular status (it is one of the simplest and most popular examples of Calderón–Zygmund operators) and its relevance in signal processing. In our approach the Hilbert transform (as well as a number of other operators) is completely expressed in terms of filter banks, which makes it easy to handle for a wide variety of scientific communities. In addition to the Hilbert transform, we construct derivative and integration operators including those of fractional order.

Our approach also allows us to consider the following related problem. In signal processing it is often useful to deal with wavelets that belong to the complex Hardy space $H^2(\mathbb{R})$, i.e., wavelets such that their Fourier transform is zero for negative frequencies. For instance, such wavelets are considered to be more efficient for the identification of "chirps" (i.e., amplitude- and frequency-modulated components) in signals. In particular, it is easy to identify the "carrier" frequency and remove it by shifting in the frequency domain if necessary. However, there does not exist any orthonormal multiresolution analysis of $H^2(\mathbb{R})$ where the associated wavelet has such a property [1]. Nevertheless, as we show in this paper, it is possible to keep the algorithmic structure of multiresolution analysis and use wavelets that approximate the Hilbert transform of a given real-valued function with any given (but finite) accuracy. The sum of the original wavelet and i times its approximate Hilbert transform yields a new wavelet that is approximately in $H^2(\mathbb{R})$. As a direct consequence, we obtain a fast algorithm for the computation of the Hilbert transform and a pyramidal algorithm for discrete wavelet transform with complex analytic (or progressive) wavelet [11]. This may be thought of as a starting point to carry on an analysis similar to that developed in [7, 4, 13].

The first part of the paper is devoted to the representation of operators in terms of filter banks. To illustrate our approach, we derive in Section II the function m_T in (1.1) for the Hilbert transform. We then present in Section III a general approach of filter bank implementation of convolution operators (in these two sections, we consider compactly supported orthogonal wavelets and obtain O(N) algorithms). We then turn to the particular case of the autocorrelation of Daubechies' compactly supported wavelets in Section IV. We again consider first the Hilbert transform and then, in Section V, develop approximations of other operators (e.g., operators of fractional differentiation and integration) by our technique.

In the second part of the paper we address the signal processing problems using autocorrelation wavelets. In Section VI we consider computing the Hilbert transform as well as for implementing the Hilbert transform,

$$(Hf)(x) = \frac{1}{\ } \text{ p.v.} \int_{-1}^{1} \frac{f(s)}{x - s}\, ds: \qquad (2.1)$$

The general case and some other operators will be considered in the following sections.

Let us start with the usual (MRA) (see Appendix). It is well known that the Hilbert transform of the wavelet (x) is still a wavelet. Since we will be interested in computing coe cients $\langle Hf; {}^{j}_{k}\rangle$, let us consider the function H given in the Fourier domain by

$$(\hat{H} \ )(\ ) = i \text{ sgn}(\ ) \ (\ )$$

$$= i \text{ sgn}(\ ) m_1 \ \frac{\ }{2} \ \frac{\ }{2} \ ; \qquad (2.2)$$

where H denotes the adjoint of H. Let us consider the 4 -periodic function

$$m_2(\ ) = i \sum_k \text{sgn}(\ + 4\ k)$$

$$m_1(\ + 4\ k) \ _{[-2\ ;2\ ]}(\ + 4\ k): \qquad (2.3)$$

Our main observation is as follows: although i where m_H is a 2 -periodic function,

$$m_H(\ ) = m_2(2\ )m_0(\ ): \qquad (2.17)$$

The coe cients $d^j_k = \langle H f; \ ^j_k\rangle$ may then be computed as

$$\langle H f; \ ^j_k\rangle \quad 2^{j=2} \int f(\ )e^{ik2^j} \ \overline{m_2(2^{j-1}\ )}$$

where

$$s^{j-1} = \frac{2^{(j-1)=2}}{2} \int f(\ )\overline{\ (2^{j-1}\ )}e^{i\ \ 2^{j-1}}\ d\ ; \qquad (3.4)$$

and the index is not necessarily an integer. Typically, we will set to be a half-integer and, if more precision is needed, we will demonstrate that can be taken to be in $2^{-N}\mathbb{Z}$. Using (3.1), we write (3.2) as

$$d^j_k = \frac{2^{j=2}}{2} \int a(\ )f(\ )\overline{m_1(2^{j-1}\ )}\ \overline{\ (2^{j-1}\ )}e^{ik2^j}\ d\ ; \quad (3.5)$$

and note that it is su cient to nd an approximation

$$a(\ )\overline{m_1(2^{j-1}\ )}\ \overline{\ (2^{j-1}\ )}e^{ik2^j}$$
$$2^{-1=2} \sum b^j_{\ -2k}\ \overline{\ (2^{j-1}\ )}e^{i\ \ 2^{j-1}}\ ;$$

Here $K(x;y) \in S'(\mathbb{R}^2)$ is the distribution kernel of T_σ, given by

$$K(x;y) = [\mathcal{F}_1^{-1}\sigma](x-y;x) = L(x-y;x); \quad (3.17)$$

where $\mathcal{F}_1$ denotes the Fourier transform with respect to the first variable. To develop our approach, we need to specify further the symbol class we are working with. We restrict ourselves to the class of the so-called Calderón–Zygmund kernels, i.e., kernels $K(x;y)$ such that

$$|\partial_x^\alpha \partial_y^\beta K(x;y)| \le \frac{C_{\alpha;\beta}}{|x-y|^{1+\alpha+\beta}}:$$

Let $f(x) \in L^2(\mathbb{R})$, and let us compute the projection of $T_\sigma f$ onto $\mathbf{W}_j$,

$$\langle T_\sigma f;\psi^j_k\rangle = 2^{-j/2}\int_{\mathbb{R}\times\mathbb{R}} L(x-y;x)f(y)\psi(2^{-j}x-k)\,dx\,dy: \quad (3.18)$$

Let us focus on the integral with respect to x first. We write

$$\int L(x-y;x)f(y)\psi(2^{-j}x-k)\,dx = \int L(x-y;k2^j)\psi(2^{-j}x-k)\,dx + R(y;j;k); \quad (3.19)$$

where $R(y;j;k)$ is some remainder. It follows from general arguments involving the vanishing moments of $\psi(x)$ that

$$|R(y;j;k)| = O(2^{M(j-1/2)}):$$

From now on, we assume that the remainder may be neglected, i.e., that we are at a sufficiently fine scale. Assuming that we may change the order of summation in (3.18), we arrive at an approximation

$$\langle T_\sigma f;\psi^j_k\rangle \approx \frac{1}{2\pi}\int \sigma(\xi;k2^j)\hat f(\xi)e^{ik2^j\xi}\,\overline{m_1(2^{j-1}\xi)}\,\overline{\hat\phi(2^{j-1}\xi)}\,d\xi: \quad (3.20)$$

Repeating considerations of Section III.1, we construct the 4π-periodic function

$$m(\xi;k;j) = \sum_{n\in\mathbb{Z}} m_1(\xi+4\pi n)\,\overline{\sigma(2^{-j+1}(\xi+4\pi n);k2^j)}\,\chi_{[-2\pi;2\pi]}(\xi+4\pi n): \quad (3.21)$$

Setting

$$m(\xi;k;j) = \frac{1}{\sqrt{2}}\sum_l b^j_{k;l}e^{il\xi/2}; \quad (3.22)$$

we obtain the Fourier coefficients of $m(\xi;k;j)$

$$b^j_{k;l} = \sum_n g_n\frac{1}{4\pi}\int_{-2\pi}^{2\pi}\overline{\sigma(2^{-j+1}\xi;k2^j)}e^{i(n-l/2)\xi}\,d\xi: \quad (3.23)$$

It is clear that the coefficients $b^j_{k;l}$ have the expected asymptotic behavior as $l\to\infty$,

$$b^j_{k;l} = O(l^{-L-1}): \quad (3.24)$$

Finally, we obtain the algorithm for computing the wavelet coefficients in (3.18),

$$\langle T_\sigma f;\psi^j_k\rangle = \sum_l \overline{b^j_{k;l}}s^{j-1}_l: \quad (3.25)$$

This expression is similar to (3.14), except that the sum is no longer a convolution. Thus, strictly speaking, the algorithm in (3.25) is not a filter bank, since filter bank algorithms are usually understood to consist of convolutions.

III.3. Connection with BCR Approach

It is reasonable to expect that a subclass of Calderón–Zygmund operators (see e.g., vol. 2 of [17]) may be implemented numerically via filter banks. Let us consider the class of symbols $S^0_{1;1}$, where $\sigma\in S^0_{1;1}$ satisfies

$$|\partial_\xi^\alpha\partial_x^\beta\sigma(\xi;x)| \le C(\alpha;\beta)(1+|\xi|)^{-\alpha}: \quad (3.26)$$

It was shown in [3] that in wavelet bases operators of this class may be represented by sparse matrices. All information is contained in the following set of coefficients:

$$\alpha^j_{kl} = \langle T\psi^j_k;\psi^j_l\rangle$$
$$\beta^j_{kl} = \langle T\psi^j_k;\phi^j_l\rangle$$
$$\gamma^j_{kl} = \langle T\phi^j_k;\psi^j_l\rangle; \quad (3.27)$$

this gives rise to the NS-form, an alternative to the S-form consisting of the elements $\langle T\psi^j_k;\psi^{j'}_{k'}\rangle$ (see [3] for more details).

To explain the relation of the filter bank approach to that using NS-form, let us consider wavelets with good localization in the Fourier domain (e.g., Battle-Lemarié wavelets), so that for a given precision we need to consider interaction between scales which are immediate neighbors. In this case we may consider the simplified S-form where only interaction between neighboring scales is taken into account.

Thus, for a given subspace $\mathbf{W}_j$, only its mappings from subspaces $\mathbf{W}_{j+1}$, $\mathbf{W}_j$, and $\mathbf{W}_{j-1}$ are significant, and these are subspaces of $\mathbf{V}_{j-2}$. In this approximation we then consider the mapping $\mathbf{V}_{j-2} \to \mathbf{W}_j$, which is exactly the one considered in

$$\hat{T}(\ ) = m_T(\ /4)\ (\ /4); \tag{3.28}$$

where

m

$d_k^j =$

$$= \frac{1}{8}\int_{-2}^{2} \sin\frac{k}{2}\quad \mathrm{sgn}(\ )\, d$$

$$-\frac{1}{8}\sum_{l=1}^{L/2} a_{2l-1}\int_{-2}^{2}$$

$$\sin\frac{k}{2}\quad \cos((2l-1)\ )\mathrm{sgn}(\ )\, d\ : \qquad (4.8)$$

The computation of the integrals in (4.8) yields (4.5).

If k is large enough, $2k-1 > 2(2l-1)$, then $1/(1-4((2l-1)/(2k-1))^2)$ may be replaced by its Taylor series, namely,

$$\frac{1}{1-4((2l-1)/(2k-1))^2} = \sum_{p=0} 2\frac{2l-1}{2k-1}^{2p} : \quad (4.9)$$

According to Lemma IX.2 of the Appendix, the sequence $\{a_{2l-1}\}$ has $L-1$ vanishing even moments, and we have

$$b_{2k-1} = \frac{-1}{(2k-1)} 2 \sum_{p=L} (2k-1)^{-2p} \sum_{l=1}^{L/2} a_{2l-1} 2^{2p}(2l-1)^{2p}$$

$$= \frac{-1}{\ }(2k-1)^{-L-1}\sum_{l=1}^{L/2} a_{2l-1}\frac{[2(2l-1)]^L}{1-(2(2l-1)/(2k-1))^2}$$

$$= O((2k-1)^{-L-1}): \qquad (4.10)$$

IV.1.2. Pyramidal Algorithm

Let us consider the approximate wavelet transform of $H f$,

$W_j f(n +$

TABLE 1

Coe cients b_{2k-1} in (4.5), for l = 2 to 6

n	L = 2	L = 4	L = 6
1	0:4244131815783876	0:4365392724806273	0:4409487600814417
2	−0:0848826363156775	−0:1131768484209033	−0:1243701630998938
3	−0:01212609090223965	−0:03968538840732975	−0:52913851209773
4	−0:004042030300746545	0:01119331467899044	0:02194767584115772
5	−0:001837286500339341	0:001469829200271469	0:007735943159323537
6	−0:00098930811556734	0:0004190010842402825	−0:001995243258287072
7	−0:0005935848693404007	0:0001606695886936418	−0:000232854476367596
8	−0:0003840843272202589	0:00007315891947052786	−0:0000585271355764204
9	−0:000262794539677021	0:00003739368944020764	−0:00001978502086783541
10	−0:0001877103854835852	0:00002079253500741326	−7:96648850858781 10^{-6}
11	−0:0001387424588356934	0:0000123326630076163	−3:616616717774846 10^{-6}
12	−0:0001054442687151276	7:699784328080609 10^{-6}	−1:794821521695973 10^{-6}
13	−0:000082012209000655	5:012630770837775 10^{-6}	−9:54786813494256 10^{-7}
14	−0:00006504416575913947	3:378917701774295 10^{-6}	−5:371888238108361 10^{-7}
15	−0:00005245497238640281	2:345812429702546 10^{-6}	−3:165738771540177 10^{-7}
16	−0:00004291770467978535	1:670310668617303 10^{-6}	−1:939965933354726 10^{-7}
17	−0:00003556038387753632	1:215739619744941 10^{-6}	−1:229261496214611 10^{-7}
18	−0:00002979383514063819	9:02084159009443 10^{-7}	−8:01852585796044 10^{-8}
19	−0:00002521016819592433	6:808447524779539 10^{-7}	−5:365206875248337 10^{-8}
20	−0:00002152087528920315	5:2171818882981	

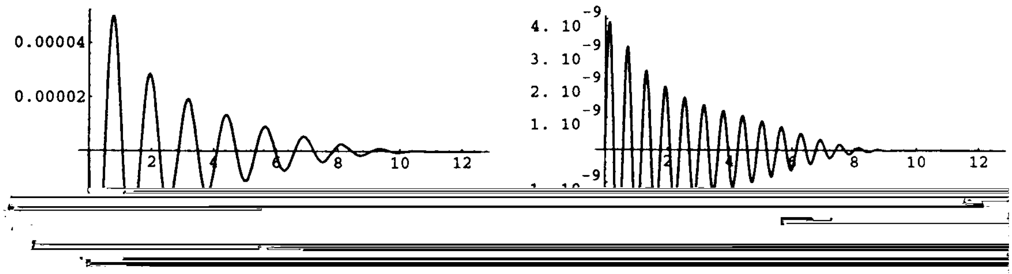

FIG. 2. Error of the approximation ( ) − ( ) for the autocorrelation of Daubechies' wavelet with $M = 2$ and $M = 5$.

$$b_k = \frac{1}{2^{n+2}} \int_{-2^n}^{2^n} \sin(2^{-n}k\ )\mathrm{sgn}(\ )\,d$$

$$- \frac{1}{2^{n+2}} \sum_{l=1}^{L/2} a_{2l-1} \int_{-2}^{2} \sin(2^{-n}k\ )$$

cos((2

of $jm_1(\ )j^2$ by the 4 -periodization of its restriction to
$[-2\ ;2$

TABLE 4

Approximate Coe cients ${}_k$ in (5.5) for the Derivative of the Autocorrelation Wavelets with 4, 5, and 6 Vanishing Moments

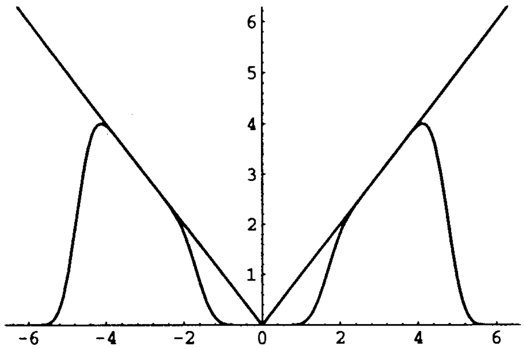

FIG. 5. Approximate filter $\mathrm{im}_5(\ )$ in (5.10) and the symbol $|\ |$ for autocorrelation of the Daubechies wavelet with 5 vanishing moments.

Using the definition of fractional derivatives

$$(\partial_x f)(x) = \int_{-\infty}^{+\infty} \frac{(x-y)_+^{-\ -1}}{\Gamma(-\ )} f(y)\,dy; \qquad (5.11)$$

where $\neq 1; 2 \ldots$ (if < 0, then (5.11) defines fractional anti-derivatives), we find its representation in the Fourier domain as

$$a(\ ) = e^{-i\ \ /2}\ _+ + e^{i\ \ /2}$$

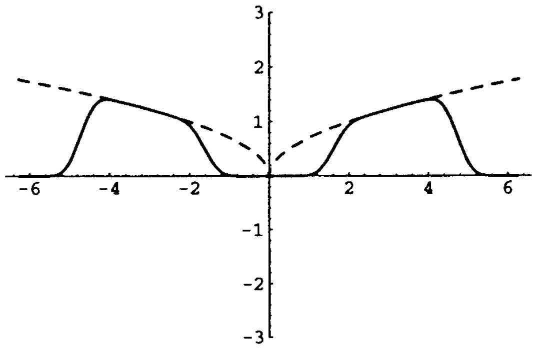

FIG. 6. Modulus of the approximation lter $m_6(\ )$ in (5.14) for the derivative of order $\frac{1}{2}$ of the Hilbert transform of autocorrelation of Daubechies' wavelet with 6 vanishing moments (dashed line, $^{1/2}$).

Battle-Lemarié wavelets or Coiflets, whose scaling function also possess vanishing moments, would do the job as well). The autocorrelation wavelets also o er an advantage of a trivial reconstruction formula (simple summation of wavelet coe cients over scales, see Eq. (9.31) in the Appendix, the price to pay being an $O(N \log(N))$ complexity).

VI.1. Bandpass Signals

Let $f \in C^r(\mathbb{R})$; $r > L$, and assume also that $Hf \in C^{r'}(\mathbb{R})$, with $r' > L$. Then, according to (9.31), we have the wavelet decompositions (we refer to (9.28) and (9.29) in the Appendix for the description of our notation)

$$f(k) = S_{j_0} f(k) + O(2^{j_0(L-1)}) \tag{6.1}$$

$$= S_J f(k) + \sum_{j=j_0+1} T_j f(k) + O(2^{j_0(L-1)}) \tag{6.2}$$

$$H(k) = S_{j_0}[Hf](k) + O(2^{j_0(L-1)}) \tag{6.3}$$

$$= S_J[Hf](k) + \sum_{j=j_0+1} T_j[Hf](k) + O(2^{j_0(L-1)}): \tag{6.4}$$

But the Hilbert transform is anti-self-adjoint,

$$T_j[Hf]($$

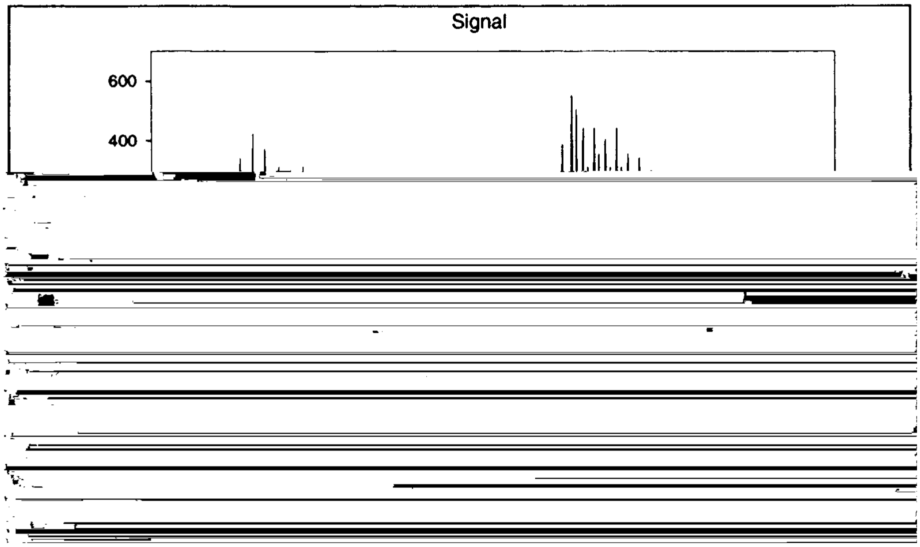

FIG. 8. An example of sound /one two/ sampled at 8 kHz.

VII. ON THE REPRESENTATION OF SIGNALS BY LOCAL PHASES AND AMPLITUDES

It is well known that an arbitrary continuous-time signal may be represented (e.g., using the method described in the previous section) in terms of its local phase or its phase derivative, the instantaneous frequency, and local amplitude (following the pioneering work of Ville [23]).

More precisely, writing

$$Z_f(x) = f(x) + i[Hf](x); \tag{7.9}$$

we obtain an analytic function (the so-called analytic signal) which may be associated with the so-called canonical pair,

$$A_f(x) = jZ_f(x)$$

lters with several vanishing moments. It is worth noting

$$0 = \int \Psi(x)\,dx = \int (2x)\,dx - \frac{1}{2}\sum_{1}^{L/2} a_{2l-1}$$

$$\int [\;(2x - 2l + 1) + \;(2x + 2l - 1)]\,dx; \qquad (9.26)$$

which implies the rst property. On the other hand, using (9.25) for $2; 4; \ldots; 2m - 2; m < L/2$, one has

$$0 = \int x^{2m}\Psi(x)\,dx$$

$$= \int x^{2m}\;(2x)\,dx - \frac{1}{2}\sum_{1}^{L/2} a_{2l-1}$$

$$\int x^{2m}[\;(2x - 2l + 1) + \;(2x + 2l - 1)]\,dx$$

$$= \int x^{2m}\;(2x)\,dx - \frac{1}{2}\sum_{1}^{L/2} a_{2l-1}$$

$$\int \Bigg[\left(x + l - \frac{1}{2}\right)^{2m} \;(2x)$$

$$+ \;\; x - l + \;1$$

12. P. G. Lemarié-Rieusset, Sur l'existence des analyses multirésolution en théorie des ondelettes, Rev. Mat. Iberoamericana **8** (1992), 457 474.

13. S. Maes, The Wavelet Transform in Signal Processing, with Applications to the Extraction of Speech Modulation Model Features, Dissertation, Université Catholique de Louvain, Louvain la Neuve, Belgium, 1994.

14. S. Mallat, Multiresolution approximation and wavelet orthonormal bases in $\mathbf{L}^2(\mathbf{R})$, Trans. Amer. Math. Soc. **315** (1989), 69 87.

15. R. J. McAulay and T. F. Quatieri, Low rate speech coding based on the sinusoidal model, in Advances in Speech Signal Processing, (S. Furui and M. Mohan Sondui, Eds.), 1992.

16. Y. Meyer, Ondelettes et fonctions splines, Technical report, sémi-